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Representations of Homotopy group through infinite symmetric product functor

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ABSTRACT

In this paper we construct symmetric product covariant functor and we study the new representations of higher order Homotopy group through infinite symmetric product covariant functor(SP^{∞}) and higher order Homotopy functor(π_n, where $n \geq 0$).

Key words: Covariant functor, Natural transformation, Homotopy Equivalance, Homotopy Group.

1. INTRODUCTION

We recall the following definitions and statements

Definition 1.1

Let X be a topological space with a base point $x_0 \in X$. For $n \geq 0$, we define the n-fold symmetric product of X, denoted by $SP^n X$ by $SP^0X = x_0$, $SP^n X = X^n/S_n$ for $n \geq 1$, where X^n denotes the n-fold Cartesian product of X with itself and S_n denotes the symmetric group on n-objects regarded as acting on X^n by permuting the coordinates.

Hence for $n \geq 1$,

$$SP^n X = \{(x_1, x_2, x_3, \ldots x_n) : x_i \in X\}$$

Definition 1.2

Let X be a topological space with a base point $x_0 \in X$. Then there is a natural continuous map

$\mu: SP^n X \rightarrow SP^{n+1} X$ defined by

$\mu(x_1, x_2, x_3, \ldots x_n) = (x_0, x_1, x_2, \ldots x_n)$,

$\Rightarrow \mu$ is a homeomorphism of $SP^n X$ onto the closed subset of $SP^{n+1} X$.

Thus we get the following sequence of spaces

$x_0 = SP^0 X \subset SP^1 X \subset \ldots\ldots \subset SP^n X \subset SP^{n+1} X \subset \ldots\ldots$,

Thus we define $\lim\limits_{n\to\infty} SP^n X = \bigcup\limits_{n=1}^{\infty} SP^n X$, is called the infinite symmetric product of X and is denoted by $SP^{\infty}X$.;

i.e., $SP^{\infty}X = \bigcup\limits_{n=1}^{\infty} SP^n X$,

Definition 1.3

A category C consists of a class of objects X,Y,...,denoted by Ob(C)

a. for each ordered pair of objects X,Y a set of morphism with domain X and range Y denoted by **C**(X,Y)

b. for each ordered triple of objects X,Y and Z and a pair of morphisms f: $X\to Y$ and g:$Y\to Z$, their composite is denoted by gf :$X\to Z$, satisfying the two axioms:
   i. associativity
   ii. identity

Definition 1.4

Let ***C*** and ***D*** be two categories. A covariant functor T from ***C*** to ***D*** consists of

a) an object function which assigns to every object X of ***C*** and object T(X) of ***D*** ; and

b) a morphism function which assigns to every morphism f: $X\to Y$ in ***C***, a morphism $T(f) : T(X) \to T(Y)$ in ***D*** such that

i) $T(I_X) = I_{T(X)}$,

ii) $T(g f) = T(g)\, T(f)$, for $g : Y\to Z$ in ***C***

Definition 1.5

Let C and D be categories. Suppose T_1 and T_2 are both covariant functors from C to D.

A natural transformation Φ from T_1 to T_2 is a function from the objects of C to the morphisms of D such that for every morphism f: $X\to Y$ in C the following condition hold:

$T_2(f)\, \Phi(X) = \Phi(Y)\, T_1(f)$

Definition 1.6

Let X be a pointed topological spaces with base point x_0 and let the set $F_n(X,x_0)$ of all continuous maps α from I^n into X for which $\alpha(\delta I^n) = x_0$.

Define a relation ~ on $F_n(X,x_0)$ as follows:

for α and β in $F_n(X,x_0)$, $\alpha \sim \beta$, if there is a homotopy $H : I^n\times I\to X$ such that $H(t_1, t_2, t_3,\dots, t_n,0) = \alpha(t_1, t_2, t_3,\dots, t_n)$,

$H(t_1, t_2, t_3,\dots, t_n, 1) = \beta(t_1, t_2, t_3,\dots, t_n)$, $(t_1, t_2, t_3,\dots, t_n)\in I^n$

$H(t_1, t_2, t_3,\dots, t_n, s) = x_0$, $(t_1, t_2, t_3,\dots, t_n)\in\delta I^n$, $s\in I$,

This relation ~ is an equivalence relation on $F_n(X,x_0)$, the equivalence class determined by α is denoted by $[\alpha]$, is called the homotopy class of α.
The set of homotopy classes of α is denoted by $\pi_n(X,x_0)$. The set of homotopy classes of α of $F_n(X,x_0)$ is a group This group is called the n-th homotopy group of X at x_0 and is denoted by $\pi_n(X,x_0)$.

Definition 1.7

A quasifibration is a map ρ:$E\to B$ such that for all $b\in B$ and $e\in\rho^{-1}(b)$,
$\rho_* : \pi_n(E, \rho^{-1}(b))\to \pi_n(B)$ is an isomorphism,where the homotopy functors are based on e and b respectively

Definition 1.8

Let X be a path connected Hausdorff pointed space and let $A \subset X$. A neighbourhood $B \supset A$ is called deformable to A if there exists a homotopy F: $X\times I \to X$ such that
$F(x,0) = x$, for all x,
$F(A\times I) \subseteq A$, $F(B\times I) \subseteq B$ and $F(B\times\{1\}) \subseteq A$.

$$\begin{tikzcd}
\pi_{n+1}(SP^{\infty}CX,SP^{\infty}X) \arrow[rr, "\partial"] \arrow[dr, "(SP^{\infty}\rho)_*"] \arrow[ddd, "(SP^{\infty}Cf)_*"'] & & \pi_n(SP^{\infty}X) \arrow[ddd, "(SP^{\infty}f)_*"] \\
& \pi_{n+1}(SP^{\infty}\Sigma X) \arrow[ur, "\partial o(SP^{\infty}\rho)^{-1}_*"'] \arrow[d, "(SP^{\infty}\Sigma f)_*"] & \\
& \pi_{n+1}(SP^{\infty}\Sigma Y) \arrow[dr, "\partial o(SP^{\infty}\rho)^{-1}_*"] & \\
\pi_{n+1}(SP^{\infty}CY,SP^{\infty}Y) \arrow[rr, "\partial"'] \arrow[ur, "(SP^{\infty}\rho)_*"] & & \pi_n(SP^{\infty}Y)
\end{tikzcd}$$

Definition 1.9

Let A and B be any two subsets of X. B is said to be deformable into A over X if the identity map I:$B\to B$ is homotopic in X to a map of B into A

Lemma 1.10

If f :$X\to Y$ and g : $X\to Y$ are such that $f \simeq g$, then

i. $SP^n f \simeq SP^n g$

ii. $SP^{\infty} f \simeq SP^{\infty} g$

Lemma 1.11

Let $\wp$ be any category and T_1 , T_2 covariant functors from $\wp$ to $\wp$. Then for any object C in $\wp$, there is an equivalence
$\theta : (T_1, T_2)\to T_2(C)$,where (T_1, T_2) is the class of natural transformations from T_1 to T_2.This Lemma is called Yoneda's Lemma.

Lemma 1.12

If f:$X\to Y$ is a homotopy equivalence between two pointed topological spaces, then $SP^n f$ and $SP^{\infty} f$ are also homotopy equivalence.

2. STUDY OF SYMMETRIC PRODUCT FUNCTOR AND HOMOTOPY FUNCTOR

In this section we study the homotopy functor π_n associated with homotopy group $\pi_n(X,x_0)$ and a symmetric product covariant functor SP^n.
To do this we use the following theorems:

Theorem 2.1 (Dold- Thom)

Let X be a Hausdorff pointed space and A be a closed path connected subspace with neighbourhood deformable to it. If ρ:$X\to X/A$ is the quotient map, then $SP^{\infty}\rho$ is a quasifibration with fibre $(SP^{\infty}\rho)^{-1}(x)$ homotopy equivalent to $SP^{\infty}A$ for all x in $SP^{\infty} X/A$

Theorem 2.2

Let X be a Hausdorff pointed space such that Y is path connected with a map f:$X\to Y$.
Let $\rho : C_f\to\Sigma X$ be the quotient map from the mapping cone of f to the suspention of X which collapses $Y\times\{0\}$ to a point. Then
$SP^{\infty}\rho : SP^{\infty} C_f \to SP^{\infty} \Sigma X$ is a quasifibration with fibre $SP^{\infty}X$

Theorem 2.3

Let p:$E\to B$ be a quasifibration. Given $b\in B$ and $e\in p^{-1}(b)$ there exists a long exact sequence
$\dots\to\pi_n(F)\to\pi_n(E)\to\pi_n(B)\to\pi_{n-1}(F)\to\dots$
where $F= p^{-1}(b)$ is the fiber and i:$F\to E$ is the inclusion.

Proposition 2.4

Let X be a path connected Hausdorff pointed space .Then for all $n \ge 0$, $\pi_n(SP^{\infty} X) \cong \pi_{n+1}(SP^{\infty} \Sigma X)$. Also this isomorphism is natural.

Proof

Applying the **Theorem 2.2** to the identity map and noting that $C_{id}\cong CX$, the long exact sequence of the quasifibration $SP^{\infty}\rho : SP^{\infty}CX \to SP^{\infty}\Sigma X$ gives
$\dots \to\pi_{n+1} (SP^{\infty}CX)\to \pi_{n+1}(SP^{\infty}\Sigma X)\to \pi_n(SP^{\infty}X)\to \pi_n(SP^{\infty}CX)\to\dots$

Since CX is contractible, $SP^{\infty}CX$ is contractible and has trivial homotopy, so this segment of the sequence reduces to
$0\to \pi_{n+1}(SP^{\infty}\Sigma X)\to \pi_n(SP^{\infty}X)\to 0$
$\Rightarrow\pi_n(SP^{\infty} X) \cong \pi_{n+1}(SP^{\infty} \Sigma X)$. In particular we have
$0\to \pi_1(SP^{\infty}\Sigma X)\to \pi_0(SP^{\infty}X)\cong 0$
$\Rightarrow 0\to \pi_1(SP^{\infty}\Sigma X)\cong 0$

Next we show that this isomorphism is natural.
Given a map f:$X\to Y$,this is equivalent to that the right square of

Theorem 2.5

Let $f : (X,x_0) \to (Y,y_0)$ be a base point preserving continuous map then

$\pi_n(f): \pi_n(X,x_0) \to \pi_n(Y,y_0)$ is a homomorphism.

Proof
The function $f_* = \pi_n(f): \pi_n(X,x_0) \to \pi_n(Y,y_0)$ be defined by
$f_*([\alpha]) = [f\alpha]$, $[\alpha] \in \pi_n(X,x_0)$.

Let $[\alpha],[\beta] \in \pi_n(X,x_0)$ then $f_*([\alpha]o[\beta]) = f_*([\alpha * \beta]) = [f(\alpha*\beta)] = [f\alpha]o[f\beta] = f_*([\alpha])o f_*([\beta])$,
Thus $f_*([\alpha]o[\beta]) = f_*([\alpha])o f_*([\beta])$
$\Rightarrow$ The function f_* is a homomorphism.

All Homotopy groups and their homomorphisms from a category , this category will be denoted by **'Hop'** then we have the following theorem:

Theorem 2.6

For all $n \geq 0$, π_n **: Top** $\to$ **Hop** is a covariant functor.

Proof
Let (X,x_0) be the pointed topological spaces in **Top**, then $\pi_n(X,x_0)$ be the homotopy group in **Hop**.

Let $f:(X,x_0) \to (Y,y_0)$ and $g: (Y,y_0) \to (Z,z_0)$ are in Top, then
$f_* = \pi_n(f): \pi_n(X,x_0) \to \pi_n(Y,y_0)$ be defined by
$f_*([\alpha]) = [f\alpha]$, $[\alpha] \in \pi_n(X,x_0)$.

Now $(g.f) : (X,x_0) \to (Z,z_0)$ in 'Top'.

Then using the **Theorem 2.5** , we have $\pi_n(g.f) = \pi_n(g)\pi_n(f)$ and $\pi_n(I_{(X,x_0)})$ $= I_{\pi_n(X,x_0)}$

Let all symmetric groups and their homomorphisms from a category, this category will be denoted by **'Sop'**.
Thus the theorem follows.

Theorem 2.7

SP^n : **Top** $\to$ **Sop** is a covariant functor, for all $n \in N$

Proof
Let $f: X \to Y$ be base point preserving continuous map and

$SP^n(f): SP^n(X) \to SP^n(Y)$ be defined by
$SP^n f(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}, \ldots, x_{\alpha(n)}) = (f(x_{\alpha(1)}), f(x_{\alpha(2)}), f(x_{\alpha(3)}), \ldots, f(x_{\alpha(n)}))$
$SP^n I(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}, \ldots, x_{\alpha(n)}) = (I(x_{\alpha(1)}), I(x_{\alpha(2)}), I(x_{\alpha(3)}), \ldots, I(x_{\alpha(n)}))$
$= (x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}, \ldots, x_{\alpha(n)})$
i.e, $SP^n I(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}, \ldots, x_{\alpha(n)}) = (x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}, \ldots, x_{\alpha(n)})$
$\Rightarrow SP^n I = Id$.

Let $f : X \to Y$ and $g : Y \to Z$ be two base point preserving continuous maps , then
$g \circ f : X \to Z$ is also base point preserving continuous map and $SP^n(g \circ f) : SP^n X \to SP^n Z$ be defined by
$SP^n (g \circ f) (x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}, \ldots, x_{\alpha(n)})$

$= ((g \circ f)(x_{\alpha(1)}), (g \circ f)(x_{\alpha(2)}), (g \circ f)(x_{\alpha(3)}), \ldots, (g \circ f)(x_{\alpha(n)}))$
$= (g(f(x_{\alpha(1)})), g(f(x_{\alpha(2)})), g(f(x_{\alpha(3)})), \ldots, g(f(x_{\alpha(n)})))$
$= SP^n g(f(x_{\alpha(1)}), f(x_{\alpha(2)}), f(x_{\alpha(3)}), \ldots, f(x_{\alpha(n)}))$
$= SP^n g(SP^n f(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}, \ldots, x_{\alpha(n)}))$

$= (SP^n g \circ SP^n f)(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}, \ldots, x_{\alpha(n)})$

Thus $SP^n(g \circ f) = SP^n g \circ SP^n f$

$\Rightarrow SP^n$ is a covariant functor

Corollary 2.8
SP^∞ is also a covariant functor

Proof
Using the **Definition1.2** and **Theorem 2.7**, it follows.

3. NEW REPRESENTATION OF HIGHER ORDER HOMOTOPY GROUP ASSOCIATED WITH INFINITE SYMMETRIC PRODUCT FUNCTORS

Theorem 3.1

Let **Top** denotes the category of pointed topological space and base point preserving continuous maps and **Hop** denotes the category of Homotopy groups and their homomorphisms, then there is an equivalence $\theta : (\pi_n, SP^n) \to \pi_n(X, x_0)$, where (π_n, SP^n) denotes the set of all natural transformations from π_n to SP^n.

Proof
Using the **Definition 1.5** and **Lemma 1.11**, it follows.

Corollary 3.2

Let **Top** denotes the category of pointed topological space and base point preserving continuous maps and **Hop** denotes the category of Homotopy groups and their homomorphisms , then there is an equivalence $\theta : (\pi_n, SP^\infty) \to \pi_n(X,x_0)$, where (π_n, SP^∞) is the set of all natural transformations from π_n to SP^∞.

Proof

Using **Theorem3.1** and using $\lim_{n\to\infty} SP^n X = \bigcup_{n=1}^{\infty} SP^n X$, it follows.

Using the Theorem 3.1, we have the following:

Theorem 3.3

Let **Top** denotes the category of pointed topological space and base point preserving continuous maps and **Hop** denotes the category of Homotopy groups and their homomorphisms ,then the cardinal number $|(\pi_n, SP^n)| = |\pi_n(X,x_0)|$, where π_n and SP^n are covariant functors ;
where $|X|$ denote the underlying set of the topological space.

Theorem 3.4

Let **Top** denotes the category of pointed topological space and base point preserving continuous maps and **Hop** denotes the category of Homotopy groups and their homomorphisms ,then the cardinal number $|(\pi_n, SP^\infty)| = |\pi_n(X,x_0)|$ where π_n and SP^∞ are covariant functors ,
where $|X|$ denote the underlying set of the topological space.

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